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# A time reversal based algorithm for solving initial data inverse problems

K. Ito\*, K. Ramdani<sup>†,‡</sup> and M. Tucsnak<sup>‡,†</sup>

## Abstract

We propose an iterative algorithm to solve initial data inverse problems for a class of linear evolution equations, including the wave, the plate, the Schrödinger and the Maxwell equations in a bounded domain  $\Omega$ . We assume that the only available information is a distributed observation (i.e. partial observation of the solution on a sub-domain  $\omega$  during a finite time interval  $(0, \tau)$ ). Under some quite natural assumptions (essentially : the exact observability of the system for some time  $\tau_{obs} > 0$ ,  $\tau \geq \tau_{obs}$  and the existence of a time-reversal operator for the problem), an iterative algorithm based on a Neumann series expansion is proposed. Numerical examples are presented to show the efficiency of the method.

## 1 Introduction

In many areas of science and engineering it is important to estimate the initial state of a system governed by partial differential equations from observations over some finite time interval. In oceanography and meteorology this problem is called *data assimilation*, see for example Auroux and Blum [2], Le Dimet, Shutyaev Gejadze [11], Teng, Zhang and Huang [15] or Zou, Navon and Le Dimet [17]. Such a problem also arises in the context of medical imaging by *impedance-acoustic tomography*; see for instance Gebauer and Scherzer [5], Kuchment and Kunyansky [10] and Hristova, Kuchment and Nguyen [7]. Numerical methods relevant in this context are given in Clason and Klibanov [4]. The estimation of the initial state can also be regarded as the main step in solving inverse source problems, see Alvez et al [1]. More recently, it has been remarked that time reversal methods can be used in the context of infinite dimensional dynamical systems to identify initial states (see Phung and Zhang [13] for the case of the Kirchhoff plate equation) and source terms (see Jonsson, Gustafsson, Weston, and de Hoop [8] for the case of the wave equation).

The aim of this work is to propose a new iterative method to compute initial data for conservative linear systems and to apply it for the wave and Schrödinger equations with locally distributed observation. The basic idea is to obtain first a “reasonable” estimate of the initial state by using a Luenberger type observer and the time reversibility of the system and then to use an iterative method to refine the approximation.

More precisely, let  $X$  and  $Y$  be two Hilbert spaces which will be identified with their duals. When no risk of confusion occurs the norms in  $X$  and  $Y$  will be simply denoted by  $\|\cdot\|$ .

Given a skew-adjoint operator  $A : \mathcal{D}(A) \longrightarrow X$  generating a  $C^0$  group of isometries  $\mathbb{T}$  on  $X$ , consider the system

$$\dot{z}(t) = Az(t), \quad z(0) = x \in X. \quad (1.1)$$

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with the observation

$$y(t) = Cz(t) \quad (1.2)$$

where  $C \in \mathcal{L}(X, Y)$  is a *bounded observation operator* for  $\mathbb{T}$ .

Given  $\tau > 0$ , let  $\Psi_\tau \in \mathcal{L}(X, L^2([0, \tau]; Y))$  be the initial state to output map defined by

$$(\Psi_\tau x)(t) = C\mathbb{T}_t x \text{ for } t \in [0, \tau], \quad x \in X. \quad (1.3)$$

We are interested in solving the inverse problem of determining the initial data  $x$  from the observation  $y$ . In other words, we want to solve the equation

$$\Psi_\tau x = y, \quad (1.4)$$

where  $y \in L^2([0, \tau]; Y)$  is supposed to be a given element of the range of  $\Psi_\tau$ . Throughout the paper, we assume that the following assumptions hold:

- **(H1)** The pair  $(A, C)$  is exactly observable in some time  $\tau_{obs} > 0$ . In other words, we assume that for every  $\tau > \tau_{obs}$ , there exists  $k_\tau$  such that

$$\int_0^\tau \|C\mathbb{T}_t z_0\|^2 dt \geq k_\tau^2 \|z_0\|^2, \quad \forall z_0 \in \mathcal{D}(A). \quad (1.5)$$

- **(H2)** There exists an operator  $\mathbf{J}_\tau \in \mathcal{L}(L^2([0, \tau]; X))$ , called a *time reversal operator* associated with the pair  $(A, C)$ , satisfying the following conditions:

$$\mathbf{J}_\tau^2 = I, \quad (1.6a)$$

$$\frac{d}{dt} \mathbf{J}_\tau \mathbb{T}_t z_0 = -\mathbf{J}_\tau \frac{d}{dt} \mathbb{T}_t z_0, \quad \forall z_0 \in X \quad (1.6b)$$

$$A\mathbf{J}_\tau \mathbb{T}_t z_0 + \mathbf{J}_\tau A\mathbb{T}_t z_0 = 0, \quad \forall z_0 \in X \quad (1.6c)$$

$$C^* C \mathbf{J}_\tau v = \mathbf{J}_\tau C^* C v, \quad \forall v \in L^2([0, \tau]; X) \quad (1.6d)$$

$$\|(\mathbf{J}_\tau v)(0)\| = \|v(\tau)\|, \quad \forall v \in C^0([0, \tau]; X). \quad (1.6e)$$

The term “time reversal operator” comes from the fact that, in the examples which will be given below, the basic ingredient of  $\mathbf{J}_\tau$  is the usual time reflection operator  $\mathcal{R}_\tau$  defined by  $\mathcal{R}_\tau v = v(\tau - \cdot)$ .

Note that by the exact observability assumption **(H1)**, equation (1.4) admits a unique solution for  $y \in \text{Ran} \Psi_\tau$  for  $\tau \geq \tau_{obs}$ . Moreover this unique solution depends continuously on  $y$ .

The basic idea of our method is to replace equation (1.4) by

$$\Phi_\tau \Psi_\tau x = \Phi_\tau y,$$

where  $\Phi_\tau \in \mathcal{L}(L^2([0, \tau]; Y), X)$  is chosen such that  $\|I - \Phi_\tau \Psi_\tau\|_{\mathcal{L}(X)} < 1$ . The solution  $x$  is then computed as the sum of a Neumann series, i.e.

$$x = \sum_{n \geq 0} (I - \Phi_\tau \Psi_\tau)^n \Phi_\tau y.$$

The operator  $\Phi_\tau$  is chosen as the input to final state map of a dissipative dynamical system associated to the pair  $(A, C)$ , see Remark 2.4 below.

## 2 Main result

Let  $\gamma > 0$  be a fixed constant and let  $y = Cz$  be an output of the system (1.1). In order to reconstruct the initial state  $x$  we define a sequence  $(s_n)_{n \in \mathbb{N}}$  in  $X$  as follows

- We first set  $s_0 = (\mathbf{J}_\tau v_0)(0)$  where  $v_0$  solves

$$\begin{cases} \dot{v}_0(t) = (A - \gamma C^* C)v_0(t) + \gamma (\mathbf{J}_\tau C^* y)(t), & t \in (0, \tau) \\ v_0(0) = 0. \end{cases}$$

- For  $n \geq 1$ , we set

$$s_n = s_{n-1} - (\mathbf{J}_\tau v_n)(0) \quad (2.7)$$

with

$$\begin{cases} \dot{v}_n(t) = (A - \gamma C^* C)v_n(t) + \gamma (\mathbf{J}_\tau C^* C z_n)(t), & t \in (0, \tau) \\ v_n(0) = 0. \end{cases}$$

and

$$\begin{cases} \dot{z}_n(t) = Az_n(t), \\ z_n(0) = s_{n-1} \end{cases}$$

The main result of this work is:

**Theorem 2.1.** *Let  $A : \mathcal{D}(A) \rightarrow X$  be a skew-adjoint operator generating a  $C^0$  group of isometries  $\mathbb{T}$  on  $X$ , let  $C \in \mathcal{L}(X, Y)$  and assume that the pair  $(A, C)$  is exactly observable in time  $\tau_{\text{obs}}$  (see assumption **(H1)**). Moreover, let  $\tau > \tau_{\text{obs}}$  and  $\mathbf{J}_\tau$  be such that assumption **(H2)** holds. Then*

$$x = \sum_{n=0}^{\infty} s_n,$$

with convergence in  $X$ . More precisely, there exists  $\delta \in (0, 1)$  such that

$$\|x - \sum_{n=0}^N s_n\| \leq \delta^{N+1} \|x\| \quad (2.8)$$

and

$$\|x - \sum_{n=0}^N s_n\| \leq \frac{\delta^{N+1}}{1 - \delta} \|y\|. \quad (2.9)$$

**Remark 2.2.** *The algorithm appearing in the above theorem can be seen as a variant of the back and forth nudging method, as proposed, for instance, in [2]. The connection between these methods, in the more general context of unbounded observation operators, is investigated in Ramdani, Tucsnak and Weiss [14].*

In order to prove the above theorem we need the following preliminary result. This result is common knowledge but, in the form needed here, we did not find it in the literature.

**Lemma 2.3.** *Let  $A : \mathcal{D}(A) \rightarrow H$  be a skew-adjoint operator generating a  $C^0$  group of isometries  $\mathbb{T}$  on  $X$  and  $C \in \mathcal{L}(X, Y)$ . Assume that  $(A, C)$  is exactly observable in time  $\tau_{\text{obs}} > 0$ . Then,  $A - \gamma C^* C$  generates an exponentially stable semigroup  $\mathbb{S}$  on  $X$  for all  $\gamma > 0$ . Moreover, for every  $\tau \geq \tau_{\text{obs}}$ , there exists  $\delta \in (0, 1)$  (depending on  $\gamma$ ) such that  $\|\mathbb{S}_\tau\| \leq \delta$ .*

*Proof.* The fact that  $A - \gamma C^* C$  generates then an exponentially stable semigroup is well-known (see, for instance, [12, Theorem 2.3]). To prove the second claim, set for  $z_0 \in \mathcal{D}(A)$  and  $t \geq 0$ :

$$z(t) = \mathbb{T}_t z_0, \quad v(t) = \mathbb{S}_t z_0.$$

Using the differential equations satisfied by  $z$  and  $v$  it follows that  $w = z - v$  satisfies

$$\dot{w} = Aw - \gamma C^* C v, \quad w(0) = 0,$$

and thus

$$w(t) = -\gamma \int_0^t \mathbb{S}_{t-s} C^* C v(s) ds.$$

As  $\mathbb{S}$  is a contraction semigroup, we immediately get from the above relation that

$$\|w\|_{C([0,\tau],X)} \leq \gamma \|C^* C v\|_{L^1([0,\tau],X)},$$

from which it is easy to get that

$$\|Cw\|_{L^2([0,\tau],Y)} \leq \gamma \tau \|C\| \|C^*\| \|Cv\|_{L^2([0,\tau],Y)}.$$

Since  $Cz = Cv + Cw$ , the above relation implies that

$$\begin{aligned} \|Cz\|_{L^2([0,\tau],Y)}^2 &\leq 2 \left( \|Cv\|_{L^2([0,\tau],Y)}^2 + \|Cw\|_{L^2([0,\tau],Y)}^2 \right) \\ &\leq 2 \left( 1 + \gamma^2 \tau^2 \|C\|^2 \|C^*\|^2 \right) \|Cv\|_{L^2([0,\tau],Y)}^2 \end{aligned}$$

Combining the above inequality with the observation inequality (1.5) shows that the pair  $(A - \gamma C^* C, C)$  is exactly observable in time  $\tau_{obs}$ , as for every  $\tau \geq \tau_{obs}$  and every  $z_0 \in \mathcal{D}(A)$ :

$$\int_0^\tau \|Cv(t)\|^2 dt \geq K_\tau^2 \|z_0\|^2, \quad (2.10)$$

with the observability constant  $K_\tau = \frac{k_\tau}{\sqrt{1 + \gamma^2 \tau^2 \|C\|^2 \|C^*\|^2}}$ . Since

$$\dot{v}(t) = A - \gamma C^* C v(t), \quad v(0) = z_0,$$

it follows (taking the inner product of the first of the above equation by  $v(t)$  and integrating with respect to time) that

$$\|z_0\|^2 - \|v(\tau)\|^2 = 2 \int_0^\tau \|Cv(t)\|^2 dt.$$

This relation and (2.10) shows that  $\|\mathbb{S}_\tau\| \leq \delta := \sqrt{1 - 2K_\tau^2}$ .  $\square$

We can now prove Theorem 2.1.

*Proof of Theorem 2.1.* Let  $\Phi_\tau \in \mathcal{L}(L^2([0,\tau]; Y), X)$  be defined for all  $\xi \in L^2([0,\tau]; Y)$  by

$$\Phi_\tau \xi = (\mathbf{J}_\tau v)(0) \quad (2.11)$$

where  $v$  denotes the solution of

$$\begin{cases} \dot{v}(t) = (A - \gamma C^* C)v(t) + \gamma (\mathbf{J}_\tau C^* \xi)(t), & t \in (0, \tau) \\ v(0) = 0. \end{cases} \quad (2.12)$$

With this notation, the sequence  $(s_n)_{n \in \mathbb{N}}$  is then simply given by

$$\begin{cases} s_0 = \Phi_\tau y, \\ s_n = s_{n-1} - \Phi_\tau \Psi_\tau s_{n-1}, \quad \forall n \geq 1, \end{cases}$$

or equivalently

$$s_n = (I - \Phi_\tau \Psi_\tau)^n \Phi_\tau y, \quad \forall n \geq 0.$$

At this stage we introduce the operator  $\mathbf{F}_\tau := \Phi_\tau \Psi_\tau \in \mathcal{L}(X)$  and we claim that

$$\|I - \mathbf{F}_\tau\| \leq \delta, \quad (2.13)$$

for every  $\tau \geq \tau_{obs}$ , where  $\delta \in (0, 1)$  is the constant in Lemma 2.3. Indeed, for  $z_0 \in X$  we set

$$\begin{cases} \xi = \Psi_\tau z_0 = C \mathbb{T}_t z_0, \\ e = v - \mathbf{J}_\tau \mathbb{T}_t z_0. \end{cases}$$

Then, using (1.6b), (1.6c) and (1.6d), we immediately obtain that

$$\dot{e} = (A - \gamma C^* C)v + \gamma C^* C \mathbf{J}_\tau \mathbb{T}_t z_0 - A(\mathbf{J}_\tau \mathbb{T}_t z_0) = (A - \gamma C^* C)e.$$

Applying Lemma 2.3, there exists  $\delta \in (0, 1)$  such that

$$\|e(\tau)\| \leq \delta \|e(0)\|. \quad (2.14)$$

But, by using (2.12), (1.6e) and the fact that  $\mathbb{T}_t$  is a group of isometries, we have

$$\|e(0)\| = \|(\mathbf{J}_\tau \mathbb{T}_t z_0)(0)\| = \|(\mathbb{T}_t z_0)(\tau)\| = \|z_0\|.$$

On the other hand, using once again (1.6e), we have

$$\|e(\tau)\| = \|(\mathbf{J}_\tau e)(0)\| = \|(\mathbf{J}_\tau v)(0) - z_0\| = \|\Phi_\tau \xi - z_0\| = \|(\mathbf{F}_\tau - I)z_0\|.$$

Using the last two equalities in (2.14), and using the fact that  $z_0 \in X$  is arbitrary, we obtain (2.13).

To conclude we notice that from (2.13) it follows that  $\mathbf{F}_\tau = I - (I - \mathbf{F}_\tau)$  is invertible, with inverse

$$\mathbf{F}_\tau^{-1} = \sum_{n=0}^{\infty} (I - \mathbf{F}_\tau)^n.$$

Applying  $\Phi_\tau$  to both sides of (1.4), we get that  $x$  satisfies

$$\mathbf{F}_\tau x = \Phi_\tau y. \quad (2.15)$$

Thus,  $x$  is given by the Neumann's series expansion

$$x = \sum_{n=0}^{\infty} (I - \mathbf{F}_\tau)^n \Phi_\tau y = \sum_{n=0}^{\infty} s_n. \quad (2.16)$$

where the convergence holds in  $X$ . The error estimates (2.8) and (2.9) follow immediately from (2.13) thanks to the relations

$$x - \sum_{n=0}^N s_n = (I - \mathbf{F}_\tau)^{N+1} x = (I - \mathbf{F}_\tau)^{N+1} (\mathbf{F}_\tau)^{-1} \Phi_\tau y.$$

□

**Remark 2.4.** *The system (2.12) can be seen as an initial state observer (of Luenberger type) for the system (1.1), (1.2). Indeed, as shown in the above proof*

$$v(\tau) - x = S_\tau x,$$

where  $S_\tau$  is the semigroup generated by  $A - \gamma C^* C$ . Since, according to Lemma 2.3, the semigroup  $S_\tau$  is exponentially stable, it follows that there exist positive constants  $M, \omega$  such that

$$\|v(\tau) - x\| \leq M e^{-\omega\tau} \|x\|.$$

In other words, if  $\tau$  is large enough, the first step of the iterative method described in Theorem 2.1 provides already a good approximation of  $x$ .

Note that  $\Phi_\tau$  can be seen as the input to final state map of the initial state observer (1.2).

### 3 Examples

In this section we apply our main result to the Schrödinger and to the wave equations with locally distributed observation. We also present some numerical tests in one or two space dimensions.

Throughout this section  $d \in \mathbb{N}^*$ ,  $\Omega$  is a bounded domain of  $\mathbb{R}^d$  with smooth boundary  $\partial\Omega$  or  $\Omega$  is a rectangular domain. Moreover, recall that we denote by  $\mathcal{R}_\tau \in \mathcal{L}(L^2(0, \tau))$  the “usual” time reflection operator on  $(0, \tau)$ :

$$\mathcal{R}_\tau v = v(\tau - \cdot) \quad \forall v \in L^2(0, \tau). \quad (3.17)$$

Let  $\mathcal{O}$  be a non empty open subset of  $\Omega$  and denote by  $\chi$  the characteristic function of  $\mathcal{O}$ .

#### 3.1 The Schrödinger equation

Consider the initial and boundary value problem:

$$\begin{cases} \partial_t z(x, t) + i\Delta z(x, t) = 0, & x \in \Omega, \quad t \in (0, \tau), \\ z(x, t) = 0, & x \in \partial\Omega, \quad t \in (0, \tau), \\ z(x, 0) = z_0(x), & x \in \Omega, \end{cases}$$

with the output

$$y = z|_{\mathcal{O}}.$$

The above system fits into the abstract framework described in Section 1 if we introduce the appropriate spaces and operators. Setting  $X = L^2(\Omega)$ , we define the skew-adjoint operator

$$\begin{aligned} A : \mathcal{D}(A) &= H^2(\Omega) \cap H_0^1(\Omega) \longrightarrow X \\ A\varphi &= -i\Delta\varphi, \quad \forall \varphi \in \mathcal{D}(A), \end{aligned} \quad (3.18)$$

and we denote by  $\mathbb{T}$  the unitary group generated by  $A$ . We set  $Y = L^2(\mathcal{O})$  and the observation operator  $C \in \mathcal{L}(X, Y)$  is

$$C\varphi = \varphi|_{\mathcal{O}} \quad \forall \varphi \in X. \quad (3.19)$$

Its adjoint  $C^* \in \mathcal{L}(Y, X)$  is the extension operator by zero from  $\mathcal{O}$  to  $\Omega$ .

**Proposition 3.1.** *The operator  $\mathbf{J}_\tau \in \mathcal{L}(L^2([0, \tau]; X))$  defined by*

$$\mathbf{J}_\tau v = \overline{\mathcal{R}_\tau v} = \overline{v(\tau - \cdot)} \quad (3.20)$$

*is a time-reversal operator associated with the pair  $(A, C)$  defined by (3.18) and (3.19).*

The proof of the fact that  $\mathbb{T}$ ,  $C$  and  $\mathbf{J}_\tau$  satisfy the condition in **(H2)** is straightforward and is thus omitted.

The system  $(A, C)$  is exactly observable in any time  $\tau_{obs} > 0$ , provided that one of the following conditions holds

- $\partial\Omega$  is of class  $C^2$  and there exists  $x_0 \in \mathbb{R}^d$  such that the closure of the observation region  $\mathcal{O}$  contains the set

$$\{x \in \partial\Omega \mid (x - x_0) \cdot \nu(x) > 0\},$$

where  $\nu$  stands for the outer normal unit field to  $\partial\Omega$ .

- $\partial\Omega$  is of class  $C^\infty$  and the observation region  $\mathcal{O}$  satisfies the geometric optics condition of Bardos, Lebeau and Rauch (see [3]).
- $\Omega$  is a rectangular domain (with no restriction on  $\mathcal{O}$ ).

The first and the second assertions above follow from the corresponding property for the wave equation as shown, for instance, in Tucsnak and Weiss [16, Proposition 7.5.3]. The third assertion above has been proved in Komornik [9].

Theorem 2.1 can thus be applied for all time  $\tau > 0$ , yielding an approximation in  $L^2(\Omega)$  of the initial data through the formula

$$z_{0,N} := \sum_{n=0}^N s_n,$$

where the sequence  $(s_n)$  is defined by

- $s_0 = \overline{v_0(\tau)}$  with
$$\begin{cases} \partial_t v_0(x, t) + i\Delta v_0(x, t) + \gamma \chi(x) v_0(x, t) = \gamma \chi(x) \overline{y(x, \tau - t)}, & \Omega \times (0, \tau), \\ v_0(x, t) = 0, & \partial\Omega \times (0, \tau), \\ v_0(x, 0) = 0, & \Omega. \end{cases} \quad (3.21)$$

- For  $n \geq 1$ ,  $s_n = s_{n-1} - \overline{v_n(\tau)}$ , with

$$\begin{cases} \partial_t v_n(x, t) + i\Delta v_n(x, t) + \gamma \chi(x) v_n(x, t) = \gamma \chi(x) \overline{z_n(x, \tau - t)}, & \Omega \times (0, \tau), \\ v_n(x, t) = 0, & \partial\Omega \times (0, \tau), \\ v_n(x, 0) = 0, & \Omega, \end{cases} \quad (3.22)$$

where

$$\begin{cases} \partial_t z_n(x, t) + i\Delta z_n(x, t) = 0, & \Omega \times (0, \tau), \\ z_n(x, t) = 0, & \partial\Omega \times (0, \tau), \\ z_n(x, 0) = s_{n-1}(x), & \Omega. \end{cases} \quad (3.23)$$



### 3.2 The wave equation

Setting  $\square = \partial_{tt} - \Delta$ , we consider the initial and boundary value problem:

$$\begin{cases} \square w(x, t) = 0, & x \in \Omega, \quad t \in (0, \tau), \\ w(x, t) = 0, & x \in \partial\Omega, \quad t \in (0, \tau), \\ w(x, 0) = w^0(x), & x \in \Omega, \\ \frac{\partial w}{\partial t}(x, 0) = w^1(x), & x \in \Omega, \end{cases} \quad (3.24)$$

with the output

$$y = \dot{w}|_{\mathcal{O}}. \quad (3.25)$$

Set  $H = L^2(\Omega)$  and let us denote by  $\|\cdot\|_H$  the usual norm on  $L^2(\Omega)$ . Let  $A_0 : \mathcal{D}(A_0) \rightarrow H$  denote the positive definite and self-adjoint operator with domain  $\mathcal{D}(A_0) = H^2(\Omega) \cap H_0^1(\Omega)$  and defined by

$$A_0 \varphi = -\Delta \varphi, \quad \forall \varphi \in \mathcal{D}(A_0).$$

Then, the above evolution system can be written in the abstract form (1.1)-(1.2) provided we introduce the following notation :

- The state space  $X = \mathcal{D}(A_0^{\frac{1}{2}}) \times H$ , which is endowed with the norm defined by  $\|z\| = (\|A_0^{\frac{1}{2}} \varphi\|_H^2 + \|\psi\|_H^2)^{\frac{1}{2}}$ , for all  $z = \begin{bmatrix} \varphi \\ \psi \end{bmatrix} \in X$ .

- $A$  is the unbounded operator with domain  $\mathcal{D}(A) = \mathcal{D}(A_0) \times \mathcal{D}(A_0^{\frac{1}{2}})$  and defined by

$$A : \mathcal{D}(A) \rightarrow X, \quad A = \begin{bmatrix} 0 & I \\ -A_0 & 0 \end{bmatrix}, \quad (3.26)$$

- The output space is  $Y = L^2(\mathcal{O})$  and the observation operator  $C \in \mathcal{L}(X, Y)$  is

$$C \in \mathcal{L}(X, Y), \quad C \begin{bmatrix} \varphi \\ \psi \end{bmatrix} = \psi|_{\mathcal{O}}. \quad (3.27)$$

- The state of the system is  $z = \begin{bmatrix} w \\ \dot{w} \end{bmatrix}$ .

The operator  $A$  is clearly a skew-adjoint operator and we denote by  $\mathbb{T}$  the unitary group generated by  $A$ .

**Proposition 3.2.** *The operator  $\mathbf{J}_\tau \in \mathcal{L}(L^2([0, \tau]; H_0^1(\Omega) \times L^2(\Omega)))$  defined by*

$$\mathbf{J}_\tau = \begin{bmatrix} \mathcal{R}_\tau & 0 \\ 0 & -\mathcal{R}_\tau \end{bmatrix}, \quad (3.28)$$

where  $\mathcal{R}_\tau$  is given by (3.17), is a time-reversal operator associated with the pair  $(A, C)$  defined by (3.26) and (3.27).

*Proof.* Conditions (1.6a), (1.6b), (1.6d) and (1.6e) are clearly satisfied. Moreover, for  $z(t) = \mathbb{T}_t z_0 = \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix}$ , we have

$$A\mathbf{J}_\tau z(t) + \mathbf{J}_\tau A z(t) = \begin{bmatrix} 0 & I \\ -A_0 & 0 \end{bmatrix} \begin{bmatrix} (\mathcal{R}_\tau z_1)(t) \\ -(\mathcal{R}_\tau z_2)(t) \end{bmatrix} + \begin{bmatrix} \mathcal{R}_\tau & 0 \\ 0 & -\mathcal{R}_\tau \end{bmatrix} \begin{bmatrix} z_2(t) \\ -A_0 z_1(t) \end{bmatrix} = 0,$$

and thus property (1.6c) also holds.  $\square$

The system  $(A, C)$  is exactly observable in some time  $\tau_{obs} > 0$ , provided that one of the following conditions is satisfied

- $\partial\Omega$  is of class  $C^2$  and there exists  $x_0 \in \mathbb{R}^d$  such that the closure of the observation region  $\mathcal{O}$  contains the set

$$\{x \in \partial\Omega \mid (x - x_0) \cdot \nu(x) > 0\},$$

where  $\nu$  stands for the outer normal unit field to  $\partial\Omega$ .

- $\partial\Omega$  is of class  $C^\infty$  and the observation region  $\mathcal{O}$  satisfies the geometric optics condition of Bardos, Lebeau and Rauch.

The first assertion follows essentially from Ho [6] and the second from Bardos, Lebeau and Rauch (see [3]).

Under this condition, Theorem 2.1 shows that at the step  $N$  of the algorithm, the approximation of the initial data  $(w^0, w^1)$  is given by

$$\begin{bmatrix} w_N^0 \\ w_N^1 \end{bmatrix} = \sum_{n=0}^N s_n, \quad 0 \leq n \leq N,$$

where  $s_n = \begin{bmatrix} s_n^0 \\ s_n^1 \end{bmatrix} \in X$  is computed as follows:

- $s_0(x) = \begin{bmatrix} \varphi_0(x, \tau) \\ -\partial_t \varphi_0(x, \tau) \end{bmatrix}$  for all  $x \in \Omega$ , where  $\varphi_0$  is computed using the observation  $y$ , through the resolution of the initial boundary value problem:

$$\begin{cases} \square \varphi_0(x, t) + \gamma \chi(x) \partial_t \varphi_0(x, t) + \gamma \chi(x) y(x, \tau - t) = 0, & \Omega \times (0, \tau), \\ \varphi_0(x, t) = 0, & \partial\Omega \times (0, \tau), \\ \varphi_0(x, 0) = 0, & \Omega, \\ \partial_t \varphi_0(x, 0) = 0 & \Omega. \end{cases}$$

- For  $n \geq 1$ , we have

$$s_n(x) = s_{n-1}(x) - \begin{bmatrix} \varphi_n(x, \tau) \\ -\partial_t \varphi_n(x, \tau) \end{bmatrix}, \quad (3.29)$$

in which

$$\begin{cases} \square \varphi_n(x, t) + \gamma \chi(x) \partial_t \varphi_n(x, t) - \gamma \chi \partial_t \psi_n(x, \tau - t) = 0, & \Omega \times (0, \tau), \\ \varphi_n(x, t) = 0, & \partial\Omega \times (0, \tau), \\ \varphi_n(x, 0) = 0, & \Omega, \\ \partial_t \varphi_n(x, 0) = 0 & \Omega, \end{cases} \quad (3.30)$$

and

$$\begin{cases} \square\psi_n(x, t) = 0, & \Omega \times (0, \tau), \\ \psi_n(x, t) = 0, & \partial\Omega \times (0, \tau), \\ \psi_n(x, 0) = s_{n-1}^0(x), & \Omega, \\ \partial_t\psi_n(x, 0) = s_{n-1}^1(x) & \Omega. \end{cases} \quad (3.31)$$

## 4 Numerical results

### 4.1 The one-dimensional Schrödinger

We consider the initial data inverse problem described in subsection 3.1 in the particular case where  $\Omega = (0, 1)$  and  $\mathcal{O} = (1/3, 2/3)$ . We use a space and time finite difference scheme to discretize the initial boundary value problems (3.21), (3.22) and (3.23). More precisely, we use the implicit Crank-Nicolson scheme in time and a standard centered finite difference approximation in space. Assuming that we know  $z|_{\mathcal{O}}$  on the time interval  $(0, \tau) = (0, 0.1)$ , we would like to recover the initial data

$$z_0(x) = \sin(4\pi x) + ix(1 - x).$$

We use 100 points of discretization in space, a CFL  $\Delta t/h^2 = 0.4$  and a gain coefficient  $\gamma = 10$ . As expected the recovery of  $z_0$  can be achieved even in a very short time ( $\tau = 0.1$  here), as the system considered here is exactly observable for all  $\tau > 0$  (see Figures 1 and 2). The convergence is obtained after 15 iterations and the relative error decreases from 60% to 1% (see Figure 3), showing the efficiency of the method.

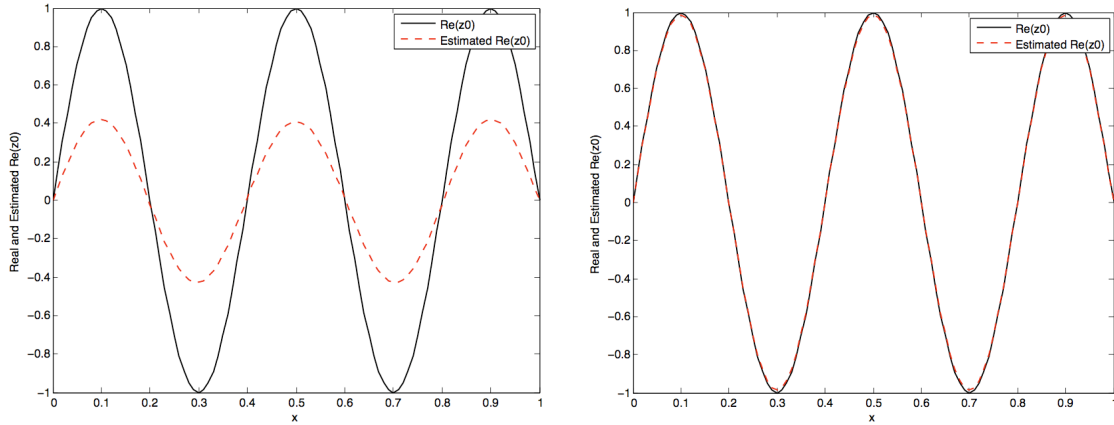


Figure 1: Exact and estimated  $\text{Re}(z_0)$  for  $\tau = 0.1$  after 1 iteration (left) and at the convergence (right).

### 4.2 The wave equation in a square

We consider the wave equation problem described in subsection 3.2 in the two-dimensional case. More precisely, let  $\Omega = (0, 1) \times (0, 1)$  denote the unit square and we want to determine an approximation of the initial data  $(w^0, w^1)$  of the initial boundary value problem (3.24)

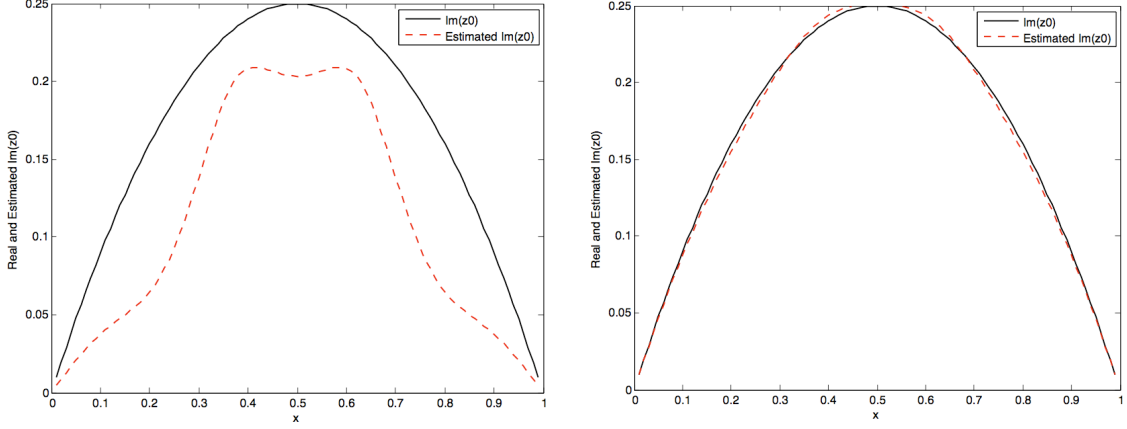


Figure 2: Exact and estimated  $\text{Im}(z_0)$  for  $\tau = 0.1$  after 1 iteration (left) and at the convergence (right).

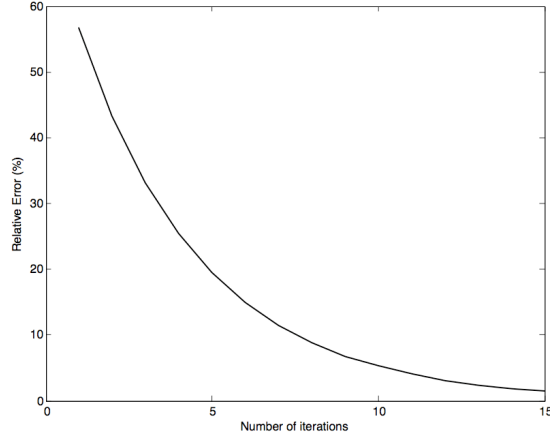


Figure 3: Relative error (in %) versus the number of iterations.

from the knowledge of the distributed observation  $y = w|_{\mathcal{O}}$  on some time interval  $(0, \tau)$ , where  $\mathcal{O}$  denotes the  $L$ -shaped region  $((0, 1/3) \times (0, 1)) \cup ((0, 1) \times (0, 1/3))$  (see Figure 4). It is clear that the above region satisfies the geometric optics condition and that the system  $(A, C)$  defined by (3.26) and (3.27) is exactly observable in time  $\tau_{obs} = 4\sqrt{2}/3$ .

We choose the following discontinuous initial data:

$$w^0(x) = \eta(x)\eta(y), \quad \eta(x) = 1/4(1 + \text{sgn}(x - 1/4))(1 + \text{sgn}(3/4 - x))$$

and

$$w_1 \equiv 0.$$

To obtain an approximation of  $(w^0, w^1)$  given  $\tau \geq \tau_{obs}$ , we apply the iterative algorithm described in subsection 3.2 using an explicit finite difference scheme to solve the initial boundary value problems (3.30) and (3.31). For the space discretization, we use a regular grid constituted of 66 discretization points in each direction. We choose the gain coefficient  $\gamma = 1$ .

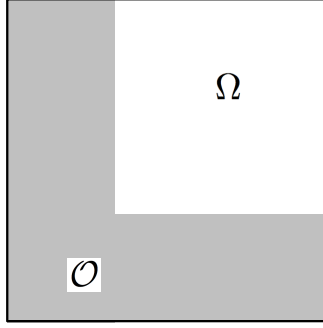


Figure 4: The observation region.

Table 1 shows the relative errors obtained and the number of iterations at the convergence for different values of  $\tau$ .

Table 1: Relative errors and number of iterations versus  $\tau$ .

$\tau/\tau_{obs}$	0.5	1	2
Relative Error on $w^0$	14.9%	3.9%	2%
Number of iterations	15	16	12

On Figure 5, we show the dependence of the relative error on  $w^0$  respect to the number of iterations for  $\tau = 2\tau_{obs}$ .

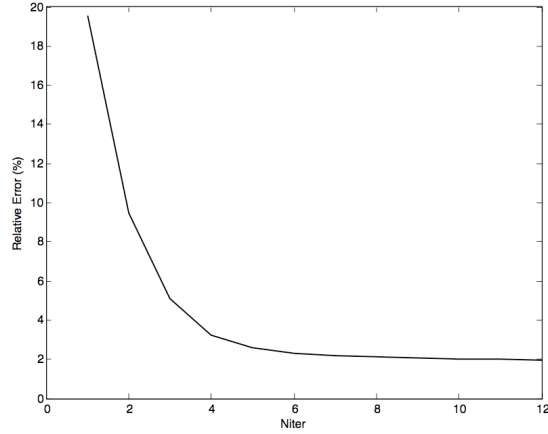


Figure 5: Relative error (in %) versus the number of iterations for  $\tau = 2\tau_{obs}$ .

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